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# A topology in a vector lattice and fixed point theorems for nonexpansive mappings (Nonlinear Analysis and Convex Analysis)

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# A topology in a vector lattice and fixed point theorems for nonexpansive mappings

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## Abstract

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

## 1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [8].

In this paper we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum  $\vee$  and the infimum  $\wedge$ , and also an order is introduced from these operators; see also [6, 9] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in case of the vector lattice with unit.

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

## 2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [2]; see also [4, 5].

Let  $X$  be a vector lattice.  $e \in X$  is said to be an unit if  $e \wedge x > 0$  for any  $x \in X$  with  $x > 0$ . Let  $\mathcal{K}_X$  be the class of units of  $X$ . In case where  $X$  is the set of real numbers  $\mathbf{R}$ ,  $\mathcal{K}_{\mathbf{R}}$  is the set of positive real numbers. Let  $X$  be a vector lattice with unit and let  $Y$  be a subset of  $X$ .  $Y$  is said to be open if for any  $x \in Y$  and for any  $e \in \mathcal{K}_X$  there exists  $\varepsilon \in \mathcal{K}_{\mathbf{R}}$  such that  $[x - \varepsilon e, x + \varepsilon e] \subset Y$ . Let  $\mathcal{O}_X$  be the class of open subsets of  $X$ .  $Y$  is said to be closed if  $Y^C \in \mathcal{O}_X$ . For  $e \in \mathcal{K}_X$  and for an interval  $[a, b]$  we consider the following subset

$$[a, b]^e = \{x \mid \text{there exists some } \varepsilon \in \mathcal{K}_{\mathbf{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of  $[a, b]^e$  it is easy to see that  $[a, b]^e \subset [a, b]$ . Every mapping from  $X \times \mathcal{K}_X$  into  $(0, \infty)$  is said to be a gauge. Let  $\Delta_X$  be the class of gauges in  $X$ . For  $x \in X$  and  $\delta \in \Delta_X$ ,  $O(x, \delta)$  is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$

$O(x, \delta)$  is said to be a  $\delta$ -neighborhood of  $x$ . Suppose that for any  $x \in X$  and for any  $\delta \in \Delta_X$  there exists  $U \in \mathcal{O}_X$  such that  $x \in U \subset O(x, \delta)$ .

For a subset  $Y$  of  $X$  we denote by  $\text{cl}(Y)$  and  $\text{int}(Y)$ , the closure and the interior of  $Y$ , respectively. Let  $X$  and  $Y$  be vector lattices with unit,  $x_0 \in Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ .  $f$  is said to be continuous in the sense of topology at  $x_0$  if for any  $V \in \mathcal{O}_Y$  with  $f(x_0) \in V$  there exists  $U \in \mathcal{O}_X$  with  $x_0 \in U$  such that  $f(U \cap Z) \subset V$ .

Let  $X$  be a vector lattice with unit.  $X$  is said to be Hausdorff if for any  $x_1, x_2 \in X$  with  $x_1 \neq x_2$  there exists  $O_1, O_2 \in \mathcal{O}_X$  such that  $x_1 \in O_1$ ,  $x_2 \in O_2$  and  $O_1 \cap O_2 = \emptyset$ . A subset  $Y$  of  $X$  is said to be compact if for any open covering of  $Y$  there exists a finite sub-covering. A subset  $Y$  of  $X$  is said to be normal if for any closed subsets  $F_1$  and  $F_2$  with  $F_1 \cap F_2 \cap Y = \emptyset$  there exists  $O_1, O_2 \in \mathcal{O}_X$  such that  $F_1 \subset O_1$ ,  $F_2 \subset O_2$  and  $O_1 \cap O_2 \cap Y = \emptyset$ .

A vector lattice is said to be Archimedean if it holds that  $x = 0$  whenever there exists  $y \in X$  with  $y \geq 0$  such that  $0 \leq rx \leq y$  for any  $r \in \mathcal{K}_{\mathbf{R}}$ .

Let  $X$  be a vector lattice with unit and  $Y$  a vector lattice,  $x_0 \in Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ .  $f$  is said to be continuous at  $x_0$  if there exists  $\{v_e \mid e \in \mathcal{K}_X\}$  satisfying the conditions (U1), (U2)<sup>d</sup> and (U3)<sup>s</sup> such that for any  $e \in \mathcal{K}_X$  there exists  $\delta \in \mathcal{K}_{\mathbf{R}}$  such that for any  $x \in Z$  if  $|x - x_0| \leq \delta e$ , then  $|f(x) - f(x_0)| \leq v_e$ ; where

$$(U1) \quad v_e \in Y \text{ with } v_e > 0;$$

$$(U2)^d \quad v_{e_1} \geq v_{e_2} \text{ if } e_1 \geq e_2;$$

$$(U3)^s \quad \text{For any } e \in \mathcal{K}_X \text{ there exists } \theta(e) \in \mathcal{K}_{\mathbf{R}} \text{ such that } v_{\theta(e)e} \leq \frac{1}{2}v_e.$$

Let  $X$  be an Archimedean vector lattice. Then there exists a positive homomorphism  $f$  from  $X$  into  $\mathbf{R}$ , that is,  $f$  satisfies the following conditions:

$$(H1) \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \text{ for any } x, y \in X \text{ and for any } \alpha, \beta \in \mathbf{R};$$

$$(H2) \quad f(x) \geq 0 \text{ for any } x \in X \text{ with } x \geq 0;$$

see [5, Example 3.1]. Suppose that there exists a homomorphism  $f$  from  $X$  into  $\mathbf{R}$  satisfying the following condition instead of (H2):

(H2)<sup>s</sup>  $f(x) > 0$  for any  $x \in X$  with  $x > 0$ .

**Example 2.1.** We consider of a sufficient condition to satisfy (H2)<sup>s</sup>. Let  $X$  be a Hilbert lattice with unit, that is,  $X$  has an inner product  $\langle \cdot, \cdot \rangle$  and for any  $x, y \in X$  if  $|x| \leq |y|$ , then  $\langle x, x \rangle \leq \langle y, y \rangle$ . For any  $e \in \mathcal{K}_X$  let  $f$  be a function from  $X$  into  $\mathbf{R}$  defined by  $f(x) = \langle x, e \rangle$ . Then  $f$  satisfies (H1) and (H2)<sup>s</sup> clearly.

### 3 Fixed point theorem for a nonexpansive mapping

Let  $X$  be a vector lattice and  $Y$  a subset of  $X$ . A mapping  $f$  from  $Y$  into  $Y$  is said to be nonexpansive if  $|f(x) - f(y)| \leq |x - y|$  for any  $x, y \in Y$ . In this section we consider a fixed point theorem for a nonexpansive mapping.

**Lemma 3.1.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $K$  a non-empty compact convex subset of  $X$ . Then*

$$c(K) = \left\{ x \mid x \in K, \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \right\}$$

*is non-empty compact convex.*

*Proof.* For any  $x \in K$  and for any  $e \in \mathcal{K}_X$  let

$$F(x, e) = \left\{ y \mid y \in K, |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e \right\}.$$

Then  $F(x, e)$  is non-empty compact convex. Let  $C(e) = \bigcap_{x \in K} F(x, e)$ . Since  $\bigcap_{i=1}^n F(x_i, e) \neq \emptyset$  for any  $x_1, \dots, x_n \in K$ ,  $C(e)$  is non-empty compact convex. Since  $C(e_1) \supset C(e_2)$  for any  $e_1, e_2 \in \mathcal{K}_X$  with  $e_1 \geq e_2$ ,  $\bigcap_{e \in \mathcal{K}_X} C(e)$  is non-empty compact convex. Moreover  $c(K) = \bigcap_{e \in \mathcal{K}_X} C(e)$ . Indeed  $c(K) \subset \bigcap_{e \in \mathcal{K}_X} C(e)$  is clear. Let  $x \in C(e)$  for any  $e \in \mathcal{K}_X$ . Then

$$|x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e$$

for any  $y \in K$ . Therefore

$$\bigvee_{y \in K} |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + \bigwedge_{e \in \mathcal{K}_X} e = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

By definition

$$\bigvee_{y \in K} |x - y| \geq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

Therefore

$$\bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|,$$

that is,  $x \in c(K)$ . □

Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $Y$  a subset of  $X$ . We say that  $Y$  has the normal structure if for any compact convex subset  $K$ , which contains two points at least, of  $Y$  there exists  $x \in K$  such that

$$\bigvee_{y \in K} |x - y| < \bigvee_{x, y \in K} |x - y|.$$

**Lemma 3.2.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $K$  a non-empty compact convex subset, which contains two points at least, of  $X$ . Suppose that  $K$  has the normal structure. Then*

$$\bigvee_{x, y \in c(K)} |x - y| < \bigvee_{x, y \in K} |x - y|.$$

*Proof.* Since  $K$  has the normal structure, there exists  $z \in K$  such that

$$|x - y| \leq \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \leq \bigvee_{y \in K} |z - y| < \bigvee_{x, y \in K} |x - y|$$

for any  $x, y \in c(K)$ . Therefore

$$\bigvee_{x, y \in c(K)} |x - y| < \bigvee_{x, y \in K} |x - y|.$$

□

**Theorem 3.3.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $K$  a non-empty compact convex subset of  $X$ . Suppose that  $K$  has the normal structure. Then every nonexpansive mapping from  $K$  into  $K$  has a fixed point.*

*Proof.* Let  $f$  be a nonexpansive mapping from  $K$  into  $K$  and  $\{K_\lambda \mid \lambda \in \Lambda\}$  the family of non-empty compact convex subsets of  $K$  satisfying that  $f(K_\lambda) \subset K_\lambda$ . By Zorn's lemma there exists a minimal element  $K_0$  of  $\{K_\lambda \mid \lambda \in \Lambda\}$ . Assume that  $K_0$  contains two points at least. By Lemma 3.1  $c(K_0)$  is non-empty compact convex. Let  $x \in c(K_0)$ . For any  $y \in K_0$

$$|f(x) - f(y)| \leq |x - y| \leq \bigvee_{y \in K_0} |x - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.$$

Let

$$M = \left\{ y \mid y \in K, |f(x) - y| \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y| \right\}.$$

Then  $f(K_0) \subset M$  and hence  $f(K_0 \cap M) \subset K_0 \cap M$ . Since  $K_0$  is a minimal element, it holds that  $K_0 \subset M$ . Therefore

$$\bigvee_{y \in K_0} |f(x) - y| \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.$$

By definition

$$\bigvee_{y \in K_0} |f(x) - y| \geq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.$$

Therefore

$$\bigvee_{y \in K_0} |f(x) - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|,$$

that is,  $f(x) \in c(K_0)$ . Since  $K_0$  is a minimal element, it holds that  $c(K_0) = K_0$  and hence

$$\bigvee_{x, y \in c(K_0)} |x - y| = \bigvee_{x, y \in K_0} |x - y|.$$

However by Lemma 3.2

$$\bigvee_{x, y \in c(K_0)} |x - y| < \bigvee_{x, y \in K_0} |x - y|.$$

It is a contradiction. Therefore  $K_0$  only contains a unique point. The point is a fixed point.  $\square$

## 4 Fixed point theorem for the commutative family of nonexpansive mappings

For any nonexpansive mapping  $f$  from  $K$  into  $K$  let  $F_K(f)$  be the set of fixed points of  $f$ .

**Lemma 4.1.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $Y$  a subset of  $X$  and  $f$  a nonexpansive mapping from  $Y$  into  $Y$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup>. Then  $F_Y(f)$  is closed.*

*Proof.* Assume that  $F_Y(f)$  is not closed. Then for any  $\delta \in \Delta_X$  there exists  $x \in F_Y(f)^C$  such that  $O(x, \delta) \not\subset F_Y(f)^C$ . Take  $y_\delta \in O(x, \delta) \cap F_Y(f)$ . Then  $f(y_\delta) = y_\delta$ . Note that every nonexpansive mapping is continuous and hence by [5, Lemma 3.2] it is also continuous in the sense of topology. Since  $\{y_\delta \mid \delta \in \Delta_X\}$  is convergent to  $x$  in the sense of topology,  $\{f(y_\delta) \mid \delta \in \Delta_X\}$  is convergent to  $f(x)$  in the sense of topology. Since  $X$  is Hausdorff,  $f(x) = x$ . It is a contradiction. Therefore  $F_Y(f)$  is closed.  $\square$

**Lemma 4.2.** *Let  $X$  be a vector lattice. If  $|x - z| = |x - w|$ ,  $|y - z| = |y - w|$  and  $|x - z| + |y - z| = |x - y|$ , then  $z = w$ .*

*Proof.* Note that  $|a + b| = |a - b|$  if and only if  $|a| \wedge |b| = 0$ . Since

$$|x - z| = \left| x - \frac{1}{2}(z + w) - \frac{1}{2}(z - w) \right|$$

and

$$|x - w| = \left| x - \frac{1}{2}(z + w) + \frac{1}{2}(z - w) \right|,$$

it holds that  $|x - \frac{1}{2}(z + w)| \wedge \frac{1}{2}|z - w| = 0$ . In the same way it holds that  $|y - \frac{1}{2}(z + w)| \wedge \frac{1}{2}|z - w| = 0$ . Note that  $(a + b) \wedge c \leq a \wedge c + b \wedge c$  for any  $a, b, c \geq 0$ . Therefore

$$\begin{aligned} |x - y| \wedge \frac{1}{2}|z - w| &\leq \left( \left| x - \frac{1}{2}(z + w) \right| + \left| y - \frac{1}{2}(z + w) \right| \right) \wedge \frac{1}{2}|z - w| \\ &\leq \left| x - \frac{1}{2}(z + w) \right| \wedge \frac{1}{2}|z - w| + \left| y - \frac{1}{2}(z + w) \right| \wedge \frac{1}{2}|z - w| \\ &= 0. \end{aligned}$$

Assume that  $z \neq w$ . Note that, if  $|b| \wedge |c| = 0$ , then  $||a| - |b|| \wedge |c| = |a| \wedge |c|$ . Therefore

$$\begin{aligned} (|x - z| + |y - z|) \wedge \frac{1}{2}|z - w| &\geq |x - z| \wedge \frac{1}{2}|z - w| \\ &\geq \left| \left| x - \frac{1}{2}(z + w) \right| - \frac{1}{2}|z - w| \right| \wedge \frac{1}{2}|z - w| \\ &= \frac{1}{2}|z - w| > 0. \end{aligned}$$

It is a contradiction. Therefore  $z = w$ . □

**Lemma 4.3.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $Y$  a subset of  $X$  and  $f$  a nonexpansive mapping from  $Y$  into  $Y$ . Then  $F_Y(f)$  is convex.*

*Proof.* Let  $x, y \in F_Y(f)$  and  $0 \leq \alpha \leq 1$ . Then

$$\begin{aligned} |x - f((1 - \alpha)x + \alpha y)| &= |f(x) - f((1 - \alpha)x + \alpha y)| \\ &\leq |x - ((1 - \alpha)x + \alpha y)| = \alpha|x - y|, \\ |y - f((1 - \alpha)x + \alpha y)| &= |f(y) - f((1 - \alpha)x + \alpha y)| \\ &\leq |y - ((1 - \alpha)x + \alpha y)| = (1 - \alpha)|x - y|. \end{aligned}$$

Since

$$\begin{aligned} |x - y| &\leq |x - f((1 - \alpha)x + \alpha y)| + |y - f((1 - \alpha)x + \alpha y)| \\ &\leq \alpha|x - y| + (1 - \alpha)|x - y| = |x - y|, \end{aligned}$$

it holds that

$$\begin{aligned} |x - f((1 - \alpha)x + \alpha y)| &= |x - ((1 - \alpha)x + \alpha y)|, \\ |y - f((1 - \alpha)x + \alpha y)| &= |y - ((1 - \alpha)x + \alpha y)|, \end{aligned}$$

and hence

$$|x - f((1 - \alpha)x + \alpha y)| + |y - f((1 - \alpha)x + \alpha y)| = |x - y|.$$

By Lemma 4.2  $f((1 - \alpha)x + \alpha y) = (1 - \alpha)x + \alpha y$ , that is,  $F_Y(f)$  is convex.  $\square$

**Theorem 4.4.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $K$  a compact convex subset of  $X$  and  $\{f_i \mid i = 1, \dots, n\}$  the finite commutative family of nonexpansive mappings from  $K$  into  $K$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup> and  $K$  has the normal structure. Then  $\bigcap_{i=1}^n F_K(f_i)$  is non-empty.*

*Proof.* Let  $\{K_\lambda \mid \lambda \in \Lambda\}$  be the family of non-empty compact convex subsets of  $K$  satisfying that  $f_i(K_\lambda) \subset K_\lambda$  for any  $i$ . By Zorn's lemma there exists a minimal element  $K_0$  of  $\{K_\lambda \mid \lambda \in \Lambda\}$ . Assume that  $K_0$  contains two points at least. By Theorem 3.3  $F_{K_0}(f_1 \circ \dots \circ f_n)$  is non-empty. Moreover by Lemma 4.1 and Lemma 4.3  $F_{K_0}(f_1 \circ \dots \circ f_n)$  is compact convex. It holds that  $f(F_{K_0}(f_1 \circ \dots \circ f_n)) = F_{K_0}(f_1 \circ \dots \circ f_n)$  for any  $i$ . It is shown as follows. Let  $x \in F_{K_0}(f_1 \circ \dots \circ f_n)$ . Since

$$f_i(x) = f_i((f_1 \circ \dots \circ f_n)(x)) = (f_1 \circ \dots \circ f_n)(f_i(x))$$

for any  $i$ ,  $f_i(x) \in F_{K_0}(f_1 \circ \dots \circ f_n)$ , that is,  $f_i(F_{K_0}(f_1 \circ \dots \circ f_n)) \subset F_{K_0}(f_1 \circ \dots \circ f_n)$ . Next let  $x_i = (f_1 \circ \dots \circ f_{i-1} \circ f_{i+1} \circ \dots \circ f_n)(x)$ . Since

$$(f_1 \circ \dots \circ f_n)(x_i) = (f_1 \circ \dots \circ f_{i-1} \circ f_{i+1} \circ \dots \circ f_n)(x) = x_i,$$

it holds that  $x_i \in F_{K_0}(f_1 \circ \dots \circ f_n)$ . Moreover  $f_i(x_i) = x$ . Therefore  $F_{K_0}(f_1 \circ \dots \circ f_n) \subset f_i(F_{K_0}(f_1 \circ \dots \circ f_n))$ . Since  $K$  has the normal structure, there exists  $x_0 \in K_0$  such that

$$\bigvee_{y \in K_0} |x_0 - y| < \bigvee_{x, y \in K_0} |x - y|.$$

Let

$$A = \left\{ x \mid x \in K_0, \bigvee_{y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x_0 - y| \right\}.$$

$A$  is non-empty and convex clearly. Moreover since  $X$  is Archimedean,  $A$  is closed and hence compact. Let  $x \in A$ . Then for any  $i$  and for any  $y \in F_{K_0}(f_1 \circ \dots \circ f_n)$

$$\begin{aligned} |f_i(x) - y| &= |f_i(x) - f_i(y_i)| \leq |x - y_i| \\ &\leq \bigvee_{y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x - y| \\ &\leq \bigvee_{y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x_0 - y| \end{aligned}$$



and hence  $f_i(a) \in A$ , that is,  $f_i(A) \subset A$ . Since  $K_0$  is minimal,  $A = K_0$ . Therefore

$$\bigvee_{x,y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x_0 - y| < \bigvee_{x,y \in F_{K_0}(f_1 \circ \dots \circ f_n)} |x - y|.$$

It is a contradiction. Therefore  $K_0$  only contains a unique point. The point is a common fixed point of  $\{f_i \mid i = 1, \dots, n\}$ .  $\square$

**Theorem 4.5.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $K$  a compact convex subset of  $X$  and  $\{f_i \mid i \in I\}$  the commutative family of nonexpansive mappings from  $K$  into  $K$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup> and  $K$  has the normal structure. Then  $\bigcap_{i \in I} F_K(f_i)$  is non-empty.*

*Proof.* By Theorem 4.4  $\bigcap_{k=1}^n F_K(f_{i_k})$  is non-empty for any finite set  $i_1, \dots, i_n \in I$ . Since  $K$  is compact,  $\bigcap_{i \in I} F_K(f_i)$  is non-empty.  $\square$

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